Econometrics 140B
Fall 2017 Midterm 1

1. Define R^2. What is true of R^2 as more regressors (covariates) are added to the model?

   Answer: R^2 measures the percentage of the variation in the dependent variable that is explained by the regression model. As more covariates are added the R^2 increases.

2. For the model

$$y = \beta_0 + \beta_1 \log(x) + u,$$

   interpret β_1.

   Answer: A 1% change in x leads to a $\frac{\beta_1}{100}$ change in y on average.

3. For the model

$$y_i = \beta_0 + \beta_1 x_i + u_i,$$

   assume that $\mathbb{E}(u_i|x_i) = 0$ and $\mathbb{E}(u_i) = 0$.

   (a) Derive the OLS estimator of β_1.

   Answer:

$$\begin{aligned}
\widehat{\beta}^{OLS} &= \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^{n} (y - x\beta_1 - \beta_0)^2 \\
\frac{\partial}{\partial \beta_0} &= \sum (-2)(y - x\beta_1 - \beta_0) = 0 \\
&\implies \sum y - \sum x\beta_1 - \sum \beta_0 = 0 \\
&\implies \sum y - \sum x\beta_1 = \sum \beta_0 \\
&\implies \sum y - \sum x\beta_1 = n\beta_0 \\
&\implies \frac{1}{n}\sum y - \frac{1}{n}\sum x\beta_1 = \beta_0 \\
&\implies \overline{y} - \beta_1 \overline{x} = \beta_0 \\
\frac{\partial}{\partial \beta_1} &= \sum (-2x)(y - x\beta_1 - \beta_0) = 0 \\
&\implies \sum yx - \sum x^2\beta_1 - \sum x\beta_0 = 0 \\
&\implies \sum yx = \beta_1 \sum x^2 - \beta_0 \sum x \\
&\implies \sum yx = \beta_1 \sum x^2 - (\overline{y} - \beta_1\overline{x}) \sum x \\
&\implies \sum yx - \overline{y}\sum x = \beta_1 \left(\sum x^2 - \overline{x} \sum x \right) \\
&\implies \widehat{\beta}_1^{OLS} = \frac{\sum yx - \overline{y}\sum x}{\sum x^2 - \overline{x}\sum x} \\
&\vdots \\
&\implies \widehat{\beta}_1^{OLS} = \frac{\sum (x - \overline{x})(y - \overline{y})}{\sum (x - \overline{x})^2}
\end{aligned}$$

(b) Prove that the OLS estimator of β_1 is unbiased.

Answer:

$$
\begin{aligned}
\mathbb{E}[\widehat{\beta}_1] &= \mathbb{E}\left[\frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}\left[\frac{\sum(x-\overline{x})(\beta_0+\beta_1 x_i + u_i - \frac{1}{n}\sum(\beta_0+\beta_1 x_i+u_i))}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}\left[\frac{\sum(x-\overline{x})(\beta_0+\beta_1 x_i + u_i - \frac{1}{n}\sum\beta_0 - \frac{1}{n}\sum\beta_1 x_i - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}\left[\frac{\sum(x-\overline{x})(\beta_0+\beta_1 x_i + u_i - \beta_0 - \beta_1\overline{x} - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}\left[\frac{\sum(x-\overline{x})(\beta_1(x_i-\overline{x}) + u_i - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}\left[\frac{\beta_1\sum(x-\overline{x})^2}{\sum(x-\overline{x})^2} + \frac{\sum(x-\overline{x})(u_i - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}\left[\beta_1 + \frac{\sum(x-\overline{x})(u_i - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\right] \\
&= \mathbb{E}[\beta_1] + \mathbb{E}\left[\frac{\sum(x-\overline{x})(u_i - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\right] \\
&= \beta_1 + \mathbb{E}\left[\frac{\sum(x-\overline{x})(u_i - \frac{1}{n}\sum u_i)}{\sum(x-\overline{x})^2}\middle| x_i\right] \\
&= \beta_1 + \frac{1}{\sum(x-\overline{x})^2}\mathbb{E}\left[\sum(x-\overline{x})(u_i - \frac{1}{n}\sum u_i)\middle| x_i\right] \\
&= \beta_1 + \frac{1}{\sum(x-\overline{x})^2}\sum\mathbb{E}\left[(x-\overline{x})(u_i - \frac{1}{n}\sum u_i)\middle| x_i\right] \\
&= \beta_1 + \frac{1}{\sum(x-\overline{x})^2}\sum(x-\overline{x})\underbrace{\mathbb{E}\left[(u_i - \frac{1}{n}\sum u_i)\middle| x_i\right]}_{0 \text{ by assumption}} \\
&= \beta_1 \qquad \therefore \widehat{\beta}_1 \text{ is an unbiased estimator of } \beta_1.
\end{aligned}
$$