Session 10: $\widehat{\beta}_1$ is an unbiased estimator of the average treatment effect in randomized experiments

Assume you are interested in the effect of a treatment (e.g. tracking, or health insurance) on the outcome (e.g. test scores, or health) of n units. Assume that only n_1 units can receive the treatment, and $n_1 < n$. Assume also that the n_1 units that will receive the treatment are randomly chosen. Let D_i be a dummy equal to 1 if unit i is randomly selected by the lottery to receive the treatment. D_i is a random variable: it is equal to 1 if individual i is randomly selected by the lottery to receive the treatment, and to 0 otherwise. Because the n_1 units that receive the treatment are chosen randomly, each unit has a probability $\frac{n_1}{n}$ of receiving the treatment: for every i in $\{1, 2, 3, ..., n\}$, $P(D_i = 1) = \frac{n_1}{n}$. Because D_i is a binary random variable, its expectation is equal to the probability it is equal to 1. Therefore, $E(D_i) = \frac{n_1}{n}$.

For every i in $\{1, 2, 3, ..., n\}$, let $y_i(1)$ denote the potential outcome of unit i with the treatment, and let $y_i(0)$ denote the potential outcome of unit i without the treatment. The $y_i(1)$s and $y_i(0)$s are real numbers, they are not random variables. For each unit, her observed outcome is $Y_i = D_i y_i(1) + (1 - D_i) y_i(0)$. In the lectures, we have assumed we are interested in the average effect of the treatment among the n_1 units that receive it. Instead, in this exercise we assume we are interested in the average effect of the treatment in the entire population, $ATE = \frac{1}{n}\sum_{i=1}^{n}(y_i(1) - y_i(0))$. It follows from the second and third properties of the summation operator that $ATE = \frac{1}{n}\sum_{i=1}^{n} y_i(1) - \frac{1}{n}\sum_{i=1}^{n} y_i(0)$.

To estimate ATE, we can use $\widehat{\beta}_1$, the coefficient of D_i in an OLS regression of Y_i on a constant and D_i. During the lectures, we have shown / will show that $\widehat{\beta}_1 = \frac{1}{n_1}\sum_{i:D_i=1} y_i(1) - \frac{1}{n-n_1}\sum_{i:D_i=0} y_i(0)$.

1) Show that $\sum_{i:D_i=1} y_i(1) = \sum_{i=1}^{n} y_i(1) W_i$ and $\sum_{i:D_i=0} y_i(0) = \sum_{i=1}^{n} y_i(0) Z_i$, where W_i and Z_i are two random variables whose formula you need to find.

Solution

$\sum_{i:D_i=1} y_i(1) = \sum_{i=1}^{n} y_i(1) D_i$. $y_i(1) D_i = 0$ for units that have $D_i = 0$. Therefore, both in the lhs and rhs summations, we are only summing the $y_i(1)$s of the units with $D_i = 1$, so these two summations are indeed equal.

$\sum_{i:D_i=0} y_i(0) = \sum_{i=1}^{n} y_i(0)(1 - D_i)$. $y_i(0)(1 - D_i) = 0$ for units that have $D_i = 1$. Therefore, both in the lhs and rhs summations, we are only summing the $y_i(0)$s of the units with $D_i = 0$, so these two summations are indeed equal.

2) Use question 1) to find a new formula for $\widehat{\beta}_1$.

Solution

$\widehat{\beta}_1 = \frac{1}{n_1}\sum_{i=1}^{n} y_i(1) D_i - \frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)(1 - D_i)$.

3) Use question 2) to prove that $\widehat{\beta}_1$ is an unbiased estimator of ATE. Hint: you need to use the fact that $E(D_i) = \frac{n_1}{n}$.

Solution

$$
\begin{aligned}
E(\widehat{\beta}_1) &= E\left(\frac{1}{n_1}\sum_{i=1}^{n} y_i(1)D_i - \frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)(1-D_i)\right) \\
&= E\left(\frac{1}{n_1}\sum_{i=1}^{n} y_i(1)D_i\right) - E\left(\frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)(1-D_i)\right) \\
&= \frac{1}{n_1}E\left(\sum_{i=1}^{n} y_i(1)D_i\right) - \frac{1}{n-n_1}E\left(\sum_{i=1}^{n} y_i(0)(1-D_i)\right) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} E\left(y_i(1)D_i\right) - \frac{1}{n-n_1}\sum_{i=1}^{n} E\left(y_i(0)(1-D_i)\right) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} y_i(1)E\left(D_i\right) - \frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)E\left(1-D_i\right) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} y_i(1)E\left(D_i\right) - \frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)\left(1-E(D_i)\right) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} y_i(1)\frac{n_1}{n} - \frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)\left(1-\frac{n_1}{n}\right) \\
&= \frac{1}{n_1}\sum_{i=1}^{n} y_i(1)\frac{n_1}{n} - \frac{1}{n-n_1}\sum_{i=1}^{n} y_i(0)\frac{n-n_1}{n} \\
&= \frac{1}{n_1}\frac{n_1}{n}\sum_{i=1}^{n} y_i(1) - \frac{1}{n-n_1}\frac{n-n_1}{n}\sum_{i=1}^{n} y_i(0) \\
&= \frac{1}{n}\sum_{i=1}^{n} y_i(1) - \frac{1}{n}\sum_{i=1}^{n} y_i(0) \\
&= ATE.
\end{aligned}
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1st equality: result from question 2. 2nd equality: P1Exp and P2Exp. 3rd equality: P2Exp. 4th equality: P3Exp. 5th equality: P2Exp. 6th equality: P1Exp and P2Exp. 7th equality: $E(D_i) = \frac{n_1}{n}$. 9th equality: P2Sum.